1. 6.1 Introduction to Linear Momentum
2. 6.2 Linear Momentum and Force
3. 6.3 Impulse
4. 6.4 Conservation of Momentum
5. 6.5 Elastic Collisions in One Dimension
6. 6.6 Inelastic Collisions in One Dimension
7. 6.7 Collisions of Point Masses in Two Dimensions

6.1 Introduction to Linear Momentum
class="introduction"

Each rugby player has great momentum, which will affect the outcome of their collisions with each other and the ground. (credit: ozzzie, Flickr)

We use the term momentum in various ways in everyday language, and most of these ways are consistent with its precise scientific definition. We speak of sports teams or politicians gaining and maintaining the momentum to win. We also recognize that momentum has something to do with

collisions. For example, looking at the rugby players in the photograph colliding and falling to the ground, we expect their momenta to have great effects in the resulting collisions. Generally, momentum implies a tendency to continue on course—to move in the same direction—and is associated with great mass and speed.

Momentum, like energy, is important because it is conserved. Only a few physical quantities are conserved in nature, and studying them yields fundamental insight into how nature works, as we shall see in our study of momentum.

6.2 Linear Momentum and Force

- Define linear momentum.
- Explain the relationship between momentum and force.
- State Newton's second law of motion in terms of momentum.
- Calculate momentum given mass and velocity.

Linear Momentum

The scientific definition of linear momentum is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. **Linear momentum** is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as

Equation:

$$\mathbf{p} = m\mathbf{v}.$$

Momentum is directly proportional to the object's mass and also its velocity. Thus the greater an object's mass or the greater its velocity, the greater its momentum. Momentum $\mathbf{p}$ is a vector having the same direction as the velocity $\mathbf{v}$. The SI unit for momentum is $\mathrm{kg} \cdot \mathrm{m/s}$.

Note:

Linear Momentum

Linear momentum is defined as the product of a system's mass multiplied by its velocity:

Equation:

$$\mathbf{p} = m\mathbf{v}.$$

Example:

Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Strategy

No information is given regarding direction, and so we can calculate only the magnitude of the momentum, p. (As usual, a symbol that is in italics is a magnitude, whereas one that is italicized, boldfaced, and has an arrow is a vector.) In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

Equation:

$$p = \mathrm{mv}$$

when only magnitudes are considered.

Solution for (a)

To determine the momentum of the player, substitute the known values for the player's mass and speed into the equation.

Equation:

$$p_{\text{player}} = (110\ \mathrm{kg})(8.00\ \mathrm{m/s}) = 880\ \mathrm{kg} \cdot \mathrm{m/s}$$

Solution for (b)

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.

Equation:

$$p_{\text{ball}} = (0.410\ \mathrm{kg})(25.0\ \mathrm{m/s}) = 10.3\ \mathrm{kg} \cdot \mathrm{m/s}$$

The ratio of the player's momentum to that of the ball is

Equation:

$$\frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9.$$

Discussion

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player's motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

Momentum and Newton's Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the "quantity of motion." Newton actually stated his **second law of motion** in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t},$$

where $\mathbf{F}_{\text{net}}$ is the net external force, $\Delta \mathbf{p}$ is the change in momentum, and Δt is the change in time.

Note:

Newton's Second Law of Motion in Terms of Momentum

The net external force equals the change in momentum of a system divided by the time over which it changes.

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}$$

Note:
Making Connections: Force and Momentum
Force and momentum are intimately related. Force acting over time can change momentum, and Newton’s second law of motion, can be stated in its most broadly applicable form in terms of momentum. Momentum continues to be a key concept in the study of atomic and subatomic particles in quantum mechanics.

This statement of Newton’s second law of motion includes the more familiar $\mathbf{F}_{\text{net}}=m\mathbf{a}$ as a special case. We can derive this form as follows. First, note that the change in momentum $\Delta\mathbf{p}$ is given by
Equation:

$$\Delta\mathbf{p}=\Delta(m\mathbf{v}).$$

If the mass of the system is constant, then
Equation:

$$\Delta(m\mathbf{v})=m\Delta\mathbf{v}.$$

So that for constant mass, Newton’s second law of motion becomes
Equation:

$$\mathbf{F}_{\text{net}}=\frac{\Delta\mathbf{p}}{\Delta t}=\frac{m\Delta\mathbf{v}}{\Delta t}.$$

Because $\frac{\Delta\mathbf{v}}{\Delta t}=\mathbf{a}$, we get the familiar equation
Equation:

$$\mathbf{F}_{\text{net}}=m\mathbf{a}$$

when the mass of the system is constant.

Newton's second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass. We will consider systems with varying mass in some detail**;** however, the relationship between momentum and force remains useful when mass is constant, such as in the following example.

Example:
Calculating Force: Venus Williams' Racquet

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}.$$

As noted above, when mass is constant, the change in momentum is given by

Equation:

$$\Delta p = m\Delta v = m(v_{\text{f}} - v_{\text{i}}).$$

In this example, the velocity just after impact and the change in time are given; thus, once Δp is calculated, $F_{\text{net}} = \frac{\Delta p}{\Delta t}$ can be used to find the force.

Solution

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.
Equation:

$$\begin{aligned}\Delta p &= m(v_{\text{f}} - v_{\text{i}}) \\ &= (0.057\text{ kg})(58\text{ m/s} - 0\text{ m/s}) \\ &= 3.306\text{ kg}\cdot\text{m/s} \approx 3.3\text{ kg}\cdot\text{m/s}\end{aligned}$$

Now the magnitude of the net external force can determined by using $F_{\text{net}} = \frac{\Delta p}{\Delta t}$:
Equation:

$$\begin{aligned}F_{\text{net}} &= \frac{\Delta p}{\Delta t} = \frac{3.306\text{ kg}\cdot\text{m/s}}{5.0\times10^{-3}\text{ s}} \\ &= 661\text{ N} \approx 660\text{ N},\end{aligned}$$

where we have retained only two significant figures in the final step.
Discussion
This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F_{\text{net}} = \text{ma}$, but one additional step would be required compared with the strategy used in this example.

Section Summary

- Linear momentum (*momentum* for brevity) is defined as the product of a system's mass multiplied by its velocity.
- In symbols, linear momentum $\mathbf{p}$ is defined to be
  Equation:

  $$\mathbf{p} = m\mathbf{v},$$

  where m is the mass of the system and $\mathbf{v}$ is its velocity.
- The SI unit for momentum is $\text{kg}\cdot\text{m/s}$.

- Newton’s second law of motion in terms of momentum states that the net external force equals the change in momentum of a system divided by the time over which it changes.
- In symbols, Newton’s second law of motion is defined to be
  Equation:

  $$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t},$$

  $\mathbf{F}_{\text{net}}$ is the net external force, $\Delta \mathbf{p}$ is the change in momentum, and Δt is the change time.

Conceptual Questions

Exercise:

Problem:

An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?

Exercise:

Problem:

An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

Exercise:

Problem:

How can a small force impart the same momentum to an object as a large force?

Problems & Exercises

Exercise:

Problem:

(a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50 m/s. (b) Compare the elephant’s momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s. (c) What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

Solution:

(a) $1.50 \times 10^4 \text{ kg} \cdot \text{m/s}$

(b) 625 to 1

(c) $6.66 \times 10^2 \text{ kg} \cdot \text{m/s}$

Exercise:

Problem:

A runaway train car that has a mass of 15,000 kg travels at a speed of 5.4 m/s down a track. Compute the time required for a force of 1500 N to bring the car to rest.

Solution:

54 s

Glossary

linear momentum
: the product of mass and velocity

second law of motion
: physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes

6.3 Impulse

- Define impulse.
- Describe effects of impulses in everyday life.
- Determine the average effective force using graphical representation.
- Calculate average force and impulse given mass, velocity, and time.

The effect of a force on an object depends on how long it acts, as well as how great the force is. In [link], a very large force acting for a short time had a great effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum $\Delta\mathbf{p}$.

By rearranging the equation $\mathbf{F}_{\text{net}} = \frac{\Delta\mathbf{p}}{\Delta t}$ to be

Equation:

$$\Delta\mathbf{p} = \mathbf{F}_{\text{net}}\Delta t,$$

we can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $\mathbf{F}_{\text{net}}\Delta t$ is given the name **impulse**. Impulse is the same as the change in momentum.

Note:

Impulse: Change in Momentum

Change in momentum equals the average net external force multiplied by the time this force acts.

Equation:

$$\Delta\mathbf{p} = \mathbf{F}_{\text{net}}\Delta t$$

The quantity $\mathbf{F}_{\text{net}}\Delta t$ is given the name impulse.

There are many ways in which an understanding of impulse can save lives, or at least limbs. The dashboard padding in a car, and certainly the airbags, allow the net force on the occupants in the car to act over a much longer time when there is a sudden stop. The momentum change is the same for an occupant, whether an air bag is deployed or not, but the force (to bring the occupant to a stop) will be much less if it acts over a larger time. Cars today have many plastic components. One advantage of plastics is their lighter weight, which results in better gas mileage. Another advantage is that a car will crumple in a collision, especially in the event of a head-on collision. A longer collision time means the force on the car will be less. Deaths during car races decreased dramatically when the rigid frames of racing cars were replaced with parts that could crumple or collapse in the event of an accident.

Bones in a body will fracture if the force on them is too large. If you jump onto the floor from a table, the force on your legs can be immense if you land stiff-legged on a hard surface. Rolling on the ground after jumping from the table, or landing with a parachute, extends the time over which the force (on you from the ground) acts.

Example:

Calculating Magnitudes of Impulses: Two Billiard Balls Striking a Rigid Wall

Two identical billiard balls strike a rigid wall with the same speed, and are reflected without any change of speed. The first ball strikes perpendicular to the wall. The second ball strikes the wall at an angle of 30° from the perpendicular, and bounces off at an angle of 30° from perpendicular to the wall.

(a) Determine the direction of the force on the wall due to each ball.

(b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Strategy for (a)

In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the x-axis to be normal to the wall and to be positive in the initial direction of motion. Choose the y-axis to be

along the wall in the plane of the second ball’s motion. The momentum direction and the velocity direction are the same.

Solution for (a)

The first ball bounces directly into the wall and exerts a force on it in the $+x$ direction. Therefore the wall exerts a force on the ball in the $-x$ direction. The second ball continues with the same momentum component in the y direction, but reverses its x-component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls is in the $-x$ direction, so the force of the wall on each ball is along the $-x$ direction.

Strategy for (b)

Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for (b)

Let u be the speed of each ball before and after collision with the wall, and m the mass of each ball. Choose the x-axis and y-axis as previously described, and consider the change in momentum of the first ball which strikes perpendicular to the wall.

Equation:

$$p_{\text{xi}} = \text{mu}; p_{\text{yi}} = 0$$

Equation:

$$p_{\text{xf}} = -\text{mu}; p_{\text{yf}} = 0$$

Impulse is the change in momentum vector. Therefore the x-component of impulse is equal to -2mu and the y-component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

Equation:

$$p_{\text{xi}} = mu \cos 30^{\text{o}}; p_{\text{yi}} = -mu \sin 30^{\text{o}}$$

Equation:

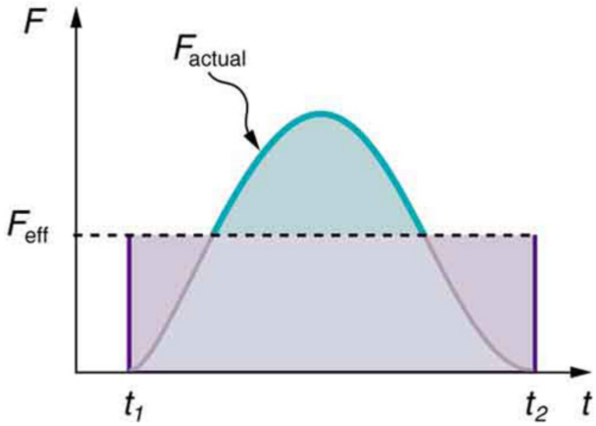
$$p_{\text{xf}} = -mu \cos 30^{\text{o}}; p_{\text{yf}} = -mu \sin 30^{\text{o}}$$

It should be noted here that while p_{x} changes sign after the collision, p_{y} does not. Therefore the x-component of impulse is equal to $-2mu$ cos 30° and the y-component of impulse is equal to zero.
The ratio of the magnitudes of the impulse imparted to the balls is
Equation:

$$\frac{2\text{mu}}{2mu\text{ cos }30^{\text{o}}}=\frac{2}{\sqrt{3}}=1.155.$$

Discussion
The direction of impulse and force is the same as in the case of (a); it is normal to the wall and along the negative x-direction. Making use of Newton's third law, the force on the wall due to each ball is normal to the wall along the positive x -direction.

Our definition of impulse includes an assumption that the force is constant over the time interval Δt. *Forces are usually not constant*. Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force F_{eff} that produces the same result as the corresponding time-varying force. [link] shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times t_1 and t_2. That area is equal to the area inside the rectangle bounded by F_{eff}, t_1, and t_2. Thus the impulses and their effects are the same for both the actual and effective forces.



A graph of force versus time with time along the x-axis and force along the y-axis for an actual force and an equivalent effective force. The areas under the two curves are equal.

Note:
Making Connections: Constant Force and Constant Acceleration
The assumption of a constant force in the definition of impulse is analogous to the assumption of a constant acceleration in kinematics. In both cases, nature is adequately described without the use of calculus.

Section Summary

- Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts:
  Equation:
  $$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t.$$
- Forces are usually not constant over a period of time.

Conceptual Questions

Exercise:

Problem: **Professional Application**

Explain in terms of impulse how padding reduces forces in a collision. State this in terms of a real example, such as the advantages of a carpeted vs. tile floor for a day care center.

Problems & Exercises

Exercise:

Problem:

A bullet is accelerated down the barrel of a gun by hot gases produced in the combustion of gun powder. What is the average force exerted on a 0.0300-kg bullet to accelerate it to a speed of 600 m/s in a time of 2.00 ms (milliseconds)?

Solution:

$9.00 \times 10^3 \text{ N}$

Exercise:

Problem: Professional Application

A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seat belt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.

Exercise:

Problem: Professional Application

A professional boxer hits his opponent with a 1000-N horizontal blow that lasts for 0.150 s. (a) Calculate the impulse imparted by this blow. (b) What is the opponent's final velocity, if his mass is 105 kg and he is motionless in midair when struck near his center of mass? (c) Calculate the recoil velocity of the opponent's 10.0-kg head if hit in this manner, assuming the head does not initially transfer significant

momentum to the boxer’s body. (d) Discuss the implications of your answers for parts (b) and (c).

Exercise:

Problem: Professional Application

One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of $4.00 \times 10^3 \text{ m/s}$, given the collision lasts $6.00 \times 10^{-8} \text{ s}$.

Exercise:

Problem: Professional Application

A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment. (a) Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm. (b) Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

Solution:

(a) $1.50 \times 10^6 \text{ N}$ away from the dashboard

(b) $1.00 \times 10^5 \text{ N}$ away from the dashboard

Exercise:

Problem:

A cruise ship with a mass of $1.00 \times 10^7 \text{ kg}$ strikes a pier at a speed of 0.750 m/s. It comes to rest 6.00 m later, damaging the ship, the pier, and the tugboat captain’s finances. Calculate the average force exerted on the pier using the concept of impulse. (Hint: First calculate the time it took to bring the ship to rest.)

Solution:

4.69×10^5 N in the boat's original direction of motion

Exercise:

Problem:

A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?

Exercise:

Problem:

When serving a tennis ball, a player hits the ball when its velocity is zero (at the highest point of a vertical toss). The racquet exerts a force of 540 N on the ball for 5.00 ms, giving it a final velocity of 45.0 m/s. Using these data, find the mass of the ball.

Solution:

60.0 g

Glossary

change in momentum
: the difference between the final and initial momentum; the mass times the change in velocity

impulse
: the average net external force times the time it acts; equal to the change in momentum

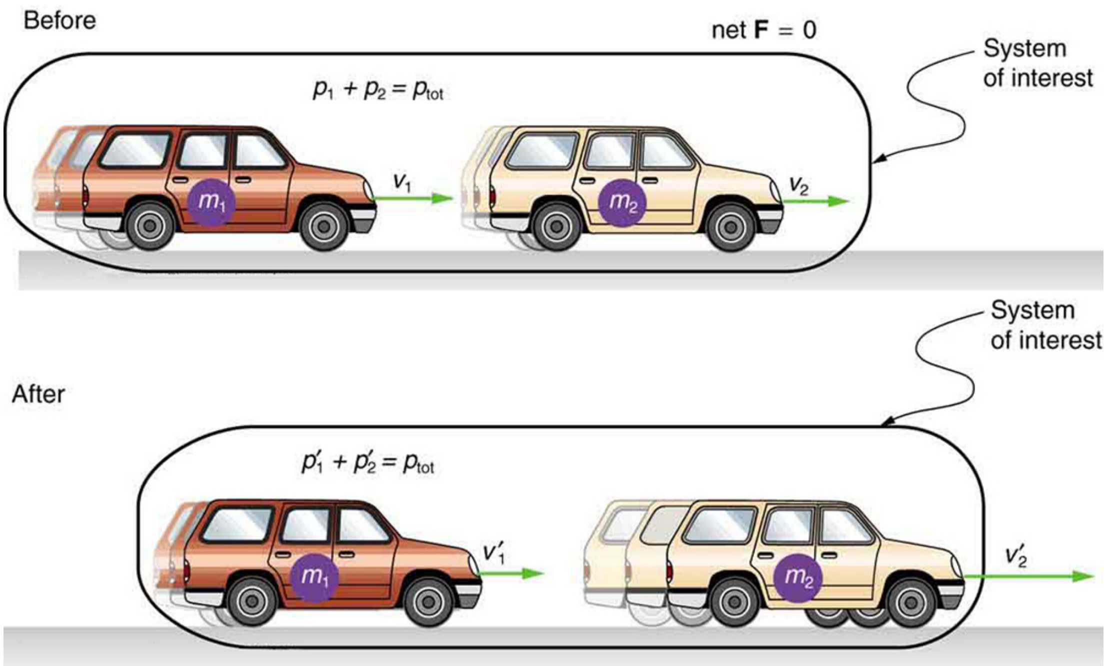
6.4 Conservation of Momentum

- Describe the principle of conservation of momentum.
- Derive an expression for the conservation of momentum.
- Explain conservation of momentum with examples.
- Explain the principle of conservation of momentum as it relates to atomic and subatomic particles.

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in Impulse and Linear Momentum and Force, where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils —conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth—for example, one car bumping into another, as shown in [link]. Both cars are coasting in the same direction when the lead car (labeled m_2) is bumped by the trailing car (labeled m_1). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible.) Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.



A car of mass m_1 moving with a velocity of v_1 bumps into another car of mass m_2 and velocity v_2 that it is following. As a result, the first car slows down to a velocity of v'_1 and the second speeds up to a velocity of v'_2. The momentum of each car is changed, but the total momentum p_{tot} of the two cars is the same before and after the collision (if you assume friction is negligible).

Using the definition of impulse, the change in momentum of car 1 is given by

Equation:

$$\Delta p_1 = F_1 \Delta t,$$

where F_1 is the force on car 1 due to car 2, and Δt is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling

near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is
Equation:

$$\Delta p_2 = F_2 \Delta t,$$

where F_2 is the force on car 2 due to car 1, and we assume the duration of the collision Δt is the same for both cars. We know from Newton's third law that $F_2 = -F_1$, and so
Equation:

$$\Delta p_2 = -F_1 \Delta t = -\Delta p_1.$$

Thus, the changes in momentum are equal and opposite, and
Equation:

$$\Delta p_1 + \Delta p_2 = 0.$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,
Equation:

$$p_1 + p_2 = \text{constant},$$

Equation:

$$p_1 + p_2 = p\prime_1 + p\prime_2,$$

where $p\prime_1$ and $p\prime_2$ are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of

objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written
Equation:

$$\mathbf{p}_{\text{tot}} = \text{constant},$$

or
Equation:

$$\mathbf{p}_{\text{tot}} = \mathbf{p'}_{\text{tot}},$$

where $\mathbf{p}_{\text{tot}}$ is the total momentum (the sum of the momenta of the individual objects in the system) and $\mathbf{p'}_{\text{tot}}$ is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero $(\mathbf{F}_{\text{net}} = 0)$.

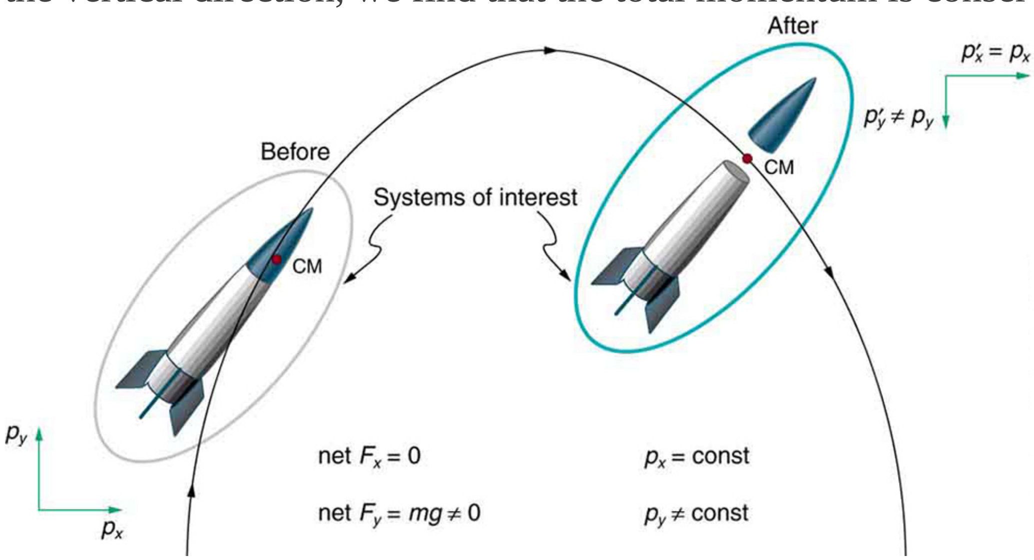
Note:
Conservation of Momentum Principle
Equation:

$$\begin{aligned} \mathbf{p}_{\text{tot}} &= \text{constant} \\ \mathbf{p}_{\text{tot}} &= \mathbf{p'}_{\text{tot}} \quad (\text{isolated system}) \end{aligned}$$

Note:
Isolated System
An isolated system is defined to be one for which the net external force is zero $(\mathbf{F}_{\text{net}} = 0)$.

Perhaps an easier way to see that momentum is conserved for an isolated system is to consider Newton's second law in terms of momentum, $\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}_{\text{tot}}}{\Delta t}$. For an isolated system, $(\mathbf{F}_{\text{net}} = 0)$; thus, $\Delta \mathbf{p}_{\text{tot}} = 0$, and $\mathbf{p}_{\text{tot}}$ is constant.

We have noted that the three length dimensions in nature—x, y, and z—are independent, and it is interesting to note that momentum can be conserved in different ways along each dimension. For example, during projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero and momentum is unchanged. But along the vertical direction, the net vertical force is not zero and the momentum of the projectile is not conserved. (See [link].) However, if the momentum of the projectile-Earth system is considered in the vertical direction, we find that the total momentum is conserved.



The horizontal component of a projectile's momentum is conserved if air resistance is negligible, even in this case where a space probe separates. The forces causing the separation are internal to the system, so that the net external horizontal force $F_{x\text{–net}}$ is still zero. The vertical component of the momentum is not conserved, because the net vertical force $F_{y\text{–net}}$ is not zero. In the vertical direction, the space probe-Earth system needs to be considered and we find that the total momentum is conserved. The center of mass of the

space probe takes the same path it would if the separation did not occur.

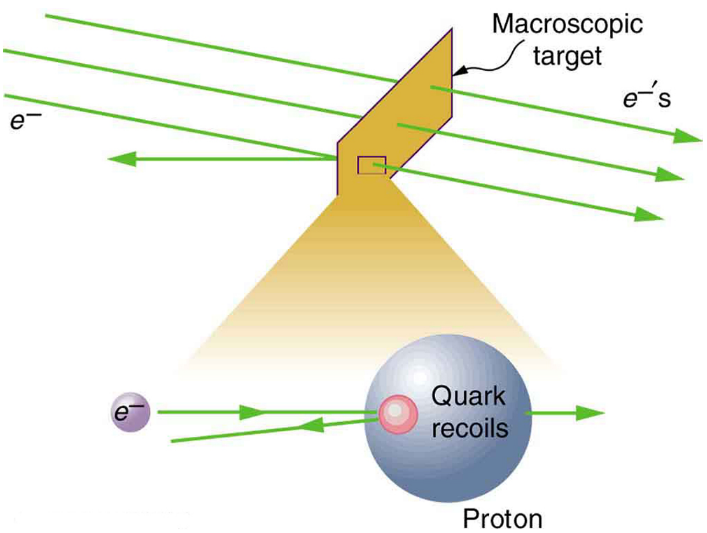
The conservation of momentum principle can be applied to systems as different as a comet striking Earth and a gas containing huge numbers of atoms and molecules. Conservation of momentum is violated only when the net external force is not zero. But another larger system can always be considered in which momentum is conserved by simply including the source of the external force. For example, in the collision of two cars considered above, the two-car system conserves momentum while each one-car system does not.

Subatomic Collisions and Momentum

The conservation of momentum principle not only applies to the macroscopic objects, it is also essential to our explorations of atomic and subatomic particles. Giant machines hurl subatomic particles at one another, and researchers evaluate the results by assuming conservation of momentum (among other things).

On the small scale, we find that particles and their properties are invisible to the naked eye but can be measured with our instruments, and models of these subatomic particles can be constructed to describe the results. Momentum is found to be a property of all subatomic particles including massless particles such as photons that compose light. Momentum being a property of particles hints that momentum may have an identity beyond the description of an object's mass multiplied by the object's velocity. Indeed, momentum relates to wave properties and plays a fundamental role in what measurements are taken and how we take these measurements. Furthermore, we find that the conservation of momentum principle is valid when considering systems of particles. We use this principle to analyze the masses and other properties of previously undetected particles, such as the nucleus of an atom and the existence of quarks that make up particles of nuclei. [link] below illustrates how a particle scattering backward from another implies that its target is massive and dense. Experiments seeking evidence that **quarks** make up protons (one type of particle that makes up

nuclei) scattered high-energy electrons off of protons (nuclei of hydrogen atoms). Electrons occasionally scattered straight backward in a manner that implied a very small and very dense particle makes up the proton—this observation is considered nearly direct evidence of quarks. The analysis was based partly on the same conservation of momentum principle that works so well on the large scale.



A subatomic particle scatters straight backward from a target particle. In experiments seeking evidence for quarks, electrons were observed to occasionally scatter straight backward from a proton.

Section Summary

- The conservation of momentum principle is written
  Equation:

$$\mathbf{p}_{\text{tot}} = \text{constant}$$

or
Equation:

$$\mathbf{p}_{\text{tot}} = \mathbf{p'}_{\text{tot}} \ \text{(isolated system)},$$

$\mathbf{p}_{\text{tot}}$ is the initial total momentum and $\mathbf{p'}_{\text{tot}}$ is the total momentum some time later.

- An isolated system is defined to be one for which the net external force is zero $(\mathbf{F}_{\text{net}} = 0)$.
- During projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero.
- Conservation of momentum applies only when the net external force is zero.
- The conservation of momentum principle is valid when considering systems of particles.

Conceptual Questions

Exercise:

Problem:

Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.

Exercise:

Problem:

Must the total energy of a system be conserved whenever its momentum is conserved? Explain why or why not.

Problems & Exercises

Exercise:

Problem: **Professional Application**

Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of 150,000 kg and a velocity of 0.300 m/s, and the second having a mass of 110,000 kg and a velocity of $-0.120\ \text{m/s}$. (The minus indicates direction of motion.) What is their final velocity?

Solution:

0.122 m/s

Exercise:

Problem:

Suppose a clay model of a koala bear has a mass of 0.200 kg and slides on ice at a speed of 0.750 m/s. It runs into another clay model, which is initially motionless and has a mass of 0.350 kg. Both being soft clay, they naturally stick together. What is their final velocity?

Exercise:

Problem: Professional Application

Consider the following question: *A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seatbelt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.* Would the answer to this question be different if the car with the 70-kg passenger had collided with a car that has a mass equal to and is traveling in the opposite direction and at the same speed? Explain your answer.

Solution:

In a collision with an identical car, momentum is conserved. Afterwards $v_{\text{f}} = 0$ for both cars. The change in momentum will be the same as in the crash with the tree. However, the force on the body is not determined since the time is not known. A padded stop will reduce injurious force on body.

Glossary

conservation of momentum principle
: when the net external force is zero, the total momentum of the system is conserved or constant

isolated system
: a system in which the net external force is zero

quark
: fundamental constituent of matter and an elementary particle

6.5 Elastic Collisions in One Dimension

- Describe an elastic collision of two objects in one dimension.
- Define internal kinetic energy.
- Derive an expression for conservation of internal kinetic energy in a one dimensional collision.
- Determine the final velocities in an elastic collision given masses and initial velocities.

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.

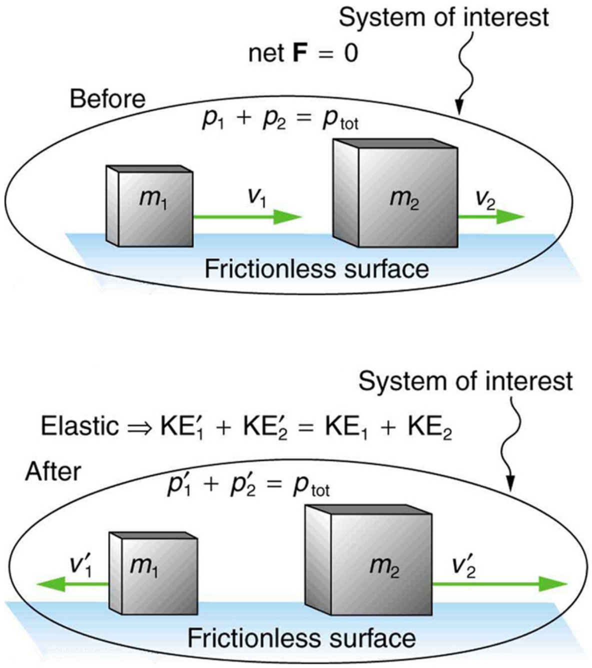
We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. [link] illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

Note:
Elastic Collision
An **elastic collision** is one that conserves internal kinetic energy.

Note:
Internal Kinetic Energy

Internal kinetic energy is the sum of the kinetic energies of the objects in the system.



An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

Equation:

$$p_1 + p_2 = p\prime_1 + p\prime_2 \quad (F_{\text{net}} = 0)$$

or

Equation:

$$m_1v_1 + m_2v_2 = m_1v\prime_1 + m_2v\prime_2 \quad (F_{\text{net}} = 0),$$

where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

Equation:

$$\frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2 = \frac{1}{2}m_1{v\prime_1}^2 + \frac{1}{2}m_2{v\prime_2}^2 \ \ \text{(two-object elastic collision)}$$

expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

Example:

Calculating Velocities Following an Elastic Collision

Calculate the velocities of two objects following an elastic collision, given that

Equation:

$$m_1 = 0.500\text{ kg},\ m_2 = 3.50\text{ kg},\ v_1 = 4.00\text{ m/s, and } \ v_2 = 0.$$

Strategy and Concept

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation shown in [link] where both objects are initially moving. We are asked to find two unknowns (the final velocities $v\prime_1$ and $v\prime_2$). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus $v_2 = 0$. Once we simplify these equations, we combine them algebraically to solve for the unknowns.

Solution

For this problem, note that $v_2 = 0$ and use conservation of momentum. Thus,

Equation:

$$p_1 = p\prime_1 + p\prime_2$$

or

Equation:

$$m_1 v_1 = m_1 v\prime_1 + m_2 v\prime_2.$$

Using conservation of internal kinetic energy and that $v_2 = 0$,

Equation:

$$\frac{1}{2} m_1 {v_1}^2 = \frac{1}{2} m_1 {v\prime_1}^2 + \frac{1}{2} m_2 {v\prime_2}^2.$$

Solving the first equation (momentum equation) for $v\prime_2$, we obtain

Equation:

$$v\prime_2 = \frac{m_1}{m_2}(v_1 - v\prime_1).$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable $v\prime_2$, leaving only $v\prime_1$ as an unknown (the algebra is left as an exercise for the reader). There are two solutions to any quadratic equation; in this example, they are

Equation:

$$v\prime_1 = 4.00 \text{ m/s}$$

and

Equation:

$$v\prime_1 = -3.00 \text{ m/s}.$$

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution $(v\prime_1 = -3.00 \text{ m/s})$ is negative, meaning that the first object bounces backward. When this negative value of $v\prime_1$ is used to find the velocity of the second object after the collision, we get

Equation:

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1) = \frac{0.500 \text{ kg}}{3.50 \text{ kg}}[4.00 - (-3.00)] \text{ m/s}$$

or

Equation:

$$v'_2 = 1.00 \text{ m/s}.$$

Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision; you will find it, too, is unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one-dimensional elastic collision of two objects. These equations can be extended to more objects if needed.

Note:

Making Connections: Take-Home Investigation—Ice Cubes and Elastic Collision

Find a few ice cubes which are about the same size and a smooth kitchen tabletop or a table with a glass top. Place the ice cubes on the surface several centimeters away from each other. Flick one ice cube toward a stationary ice cube and observe the path and velocities of the ice cubes after the collision. Try to avoid edge-on collisions and collisions with rotating ice cubes. Have you created approximately elastic collisions? Explain the speeds and directions of the ice cubes using momentum.

Note:

PhET Explorations: Collision Lab

Investigate collisions on an air hockey table. Set up your own experiments: vary the number of discs, masses and initial conditions. Is momentum

conserved? Is kinetic energy conserved? Vary the elasticity and see what happens.

Collision Lab

Section Summary

- An elastic collision is one that conserves internal kinetic energy.
- Conservation of kinetic energy and momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one dimensional two-body collisions.

Conceptual Questions

Exercise:

Problem: What is an elastic collision?

Problems & Exercises

Exercise:

Problem:

Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

Exercise:

Problem: **Professional Application**

Two manned satellites approach one another at a relative speed of 0.250 m/s, intending to dock. The first has a mass of 4.00×10^3 kg, and the second a mass of 7.50×10^3 kg. If the two satellites collide elastically rather than dock, what is their final relative velocity?

Solution:

0.250 m/s

Exercise:

Problem:

A 70.0-kg ice hockey goalie, originally at rest, catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. Suppose the goalie and the ice puck have an elastic collision and the puck is reflected back in the direction from which it came. What would their final velocities be in this case?

Glossary

elastic collision
: a collision that also conserves internal kinetic energy

internal kinetic energy
: the sum of the kinetic energies of the objects in a system

6.6 Inelastic Collisions in One Dimension

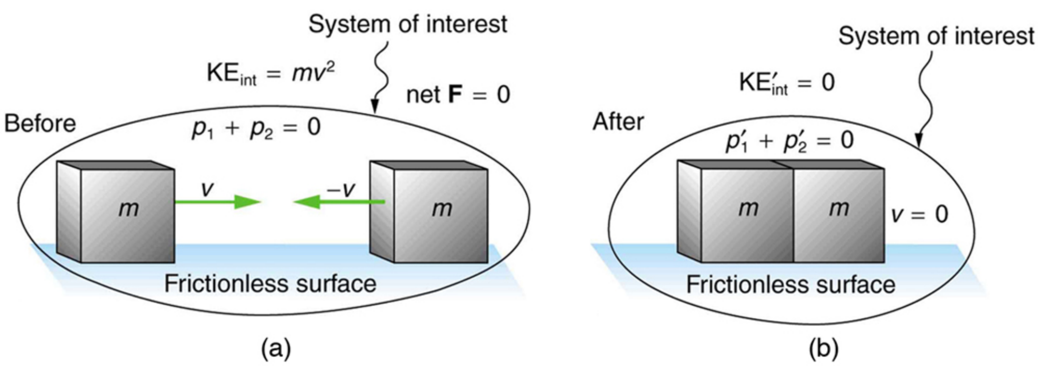
- Define inelastic collision.
- Explain perfectly inelastic collision.
- Apply an understanding of collisions to sports.
- Determine recoil velocity and loss in kinetic energy given mass and initial velocity.

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

Note:
Inelastic Collision
An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).

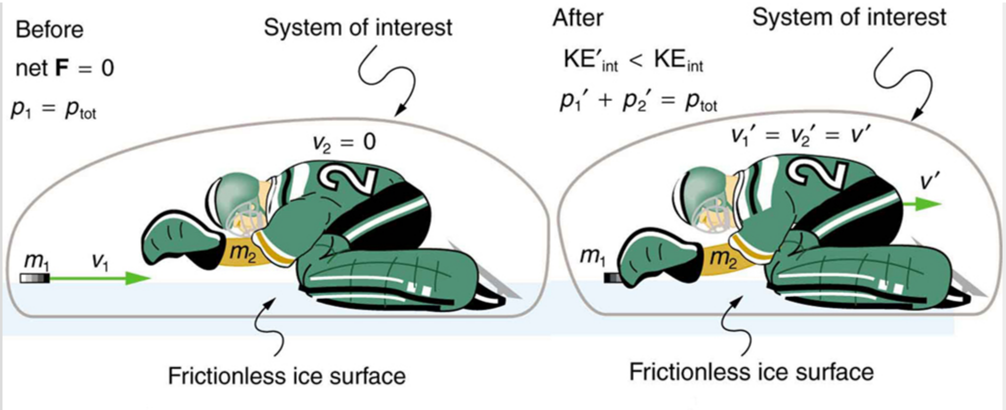
[link] shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is sometimes called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

Note:
Perfectly Inelastic Collision
A collision in which the objects stick together is sometimes called "perfectly inelastic."



An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.

Example:
Calculating Velocity and Change in Kinetic Energy: Inelastic Collision of a Puck and a Goalie
(a) Find the recoil velocity of a 70.0-kg ice hockey goalie, originally at rest, who catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. (b) How much kinetic energy is lost during the collision? Assume friction between the ice and the puck-goalie system is negligible. (See [link])



An ice hockey goalie catches a hockey puck and recoils backward. The initial kinetic energy of the puck is almost entirely converted to thermal energy and sound in this inelastic collision.

Strategy

Momentum is conserved because the net external force on the puck-goalie system is zero. We can thus use conservation of momentum to find the final velocity of the puck and goalie system. Note that the initial velocity of the goalie is zero and that the final velocity of the puck and goalie are the same. Once the final velocity is found, the kinetic energies can be calculated before and after the collision and compared as requested.

Solution for (a)

Momentum is conserved because the net external force on the puck-goalie system is zero.

Conservation of momentum is

Equation:

$$p_1 + p_2 = p\prime_1 + p\prime_2$$

or

Equation:

$$m_1 v_1 + m_2 v_2 = m_1 v\prime_1 + m_2 v\prime_2.$$

Because the goalie is initially at rest, we know $v_2 = 0$. Because the goalie catches the puck, the final velocities are equal, or $v\prime_1 = v\prime_2 = v\prime$. Thus, the

conservation of momentum equation simplifies to
Equation:

$$m_1 v_1 = (m_1 + m_2)v\prime.$$

Solving for $v\prime$ yields
Equation:

$$v\prime = \frac{m_1}{m_1 + m_2} v_1.$$

Entering known values in this equation, we get
Equation:

$$v\prime = \left(\frac{0.150 \text{ kg}}{70.0 \text{ kg} + 0.150 \text{ kg}}\right)(35.0 \text{ m/s}) = 7.48 \times 10^{-2} \text{ m/s}.$$

Discussion for (a)
This recoil velocity is small and in the same direction as the puck's original velocity, as we might expect.
Solution for (b)
Before the collision, the internal kinetic energy KE_{int} of the system is that of the hockey puck, because the goalie is initially at rest. Therefore, KE_{int} is initially
Equation:

$$\begin{aligned} \text{KE}_{\text{int}} &= \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.150 \text{ kg})(35.0 \text{ m/s})^2 \\ &= 91.9 \text{ J}. \end{aligned}$$

After the collision, the internal kinetic energy is
Equation:

$$\begin{aligned} \text{KE}\prime_{\text{int}} &= \tfrac{1}{2}(m + M)v^2 = \tfrac{1}{2}(70.15 \text{ kg})\left(7.48 \times 10^{-2} \text{ m/s}\right)^2 \\ &= 0.196 \text{ J}. \end{aligned}$$

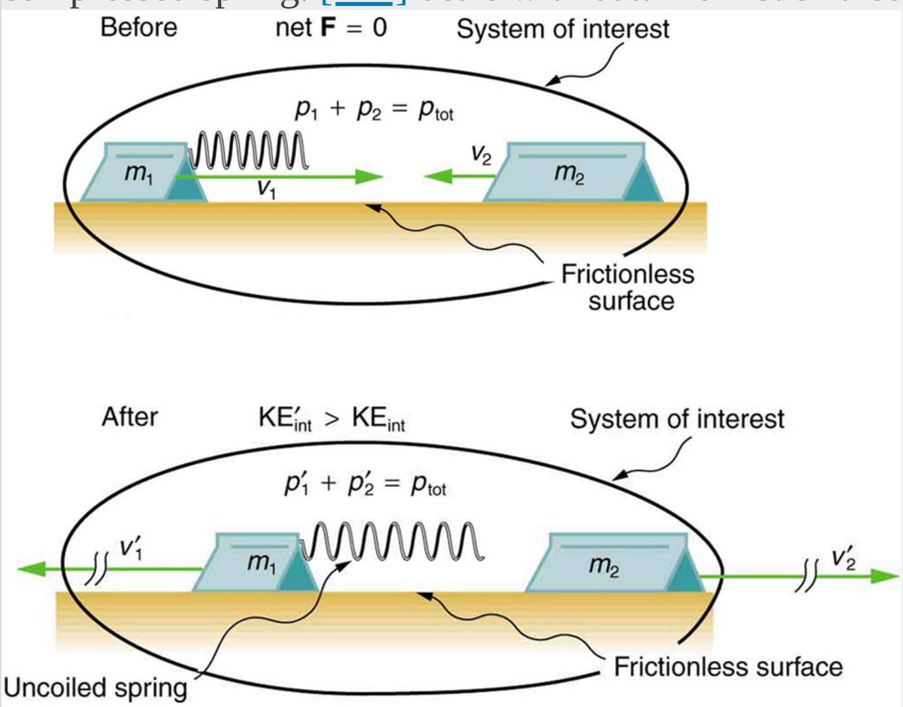
The change in internal kinetic energy is thus
Equation:

$$\begin{aligned} \mathrm{KE}'_{\mathrm{int}} - \mathrm{KE}_{\mathrm{int}} &= 0.196\ \mathrm{J} - 91.9\ \mathrm{J} \\ &= -91.7\ \mathrm{J} \end{aligned}$$

where the minus sign indicates that the energy was lost.

Discussion for (b)

Nearly all of the initial internal kinetic energy is lost in this perfectly inelastic collision. $\mathrm{KE}_{\mathrm{int}}$ is mostly converted to thermal energy and sound. During some collisions, the objects do not stick together and less of the internal kinetic energy is removed—such as happens in most automobile accidents. Alternatively, stored energy may be converted into internal kinetic energy during a collision. [link] shows a one-dimensional example in which two carts on an air track collide, releasing potential energy from a compressed spring. [link] deals with data from such a collision.



An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in [link], the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Collisions are particularly important in sports and the sporting and leisure industry utilizes elastic and inelastic collisions. Let us look briefly at tennis. Recall that in a collision, it is momentum and not force that is important. So, a heavier tennis racquet will have the advantage over a lighter one. This conclusion also holds true for other sports—a lightweight bat (such as a softball bat) cannot hit a hardball very far.

The location of the impact of the tennis ball on the racquet is also important, as is the part of the stroke during which the impact occurs. A smooth motion results in the maximizing of the velocity of the ball after impact and reduces sports injuries such as tennis elbow. A tennis player tries to hit the ball on the "sweet spot" on the racquet, where the vibration and impact are minimized and the ball is able to be given more velocity. Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Example:

Calculating Final Velocity and Energy Release: Two Carts Collide

In the collision pictured in [link], two carts collide inelastically. Cart 1 (denoted m_1 carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of $2.00\ \mathrm{m/s}$. Cart 2 (denoted m_2 in [link]) has a mass of 0.500 kg and an initial velocity of $-0.500\ \mathrm{m/s}$. After the collision, cart 1 is observed to recoil with a velocity of $-4.00\ \mathrm{m/s}$. (a) What is the final velocity of cart 2? (b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

Strategy

We can use conservation of momentum to find the final velocity of cart 2, because $F_{\text{net}} = 0$ (the track is frictionless and the force of the spring is internal). Once this velocity is determined, we can compare the internal kinetic energy before and after the collision to see how much energy was released by the spring.

Solution for (a)

As before, the equation for conservation of momentum in a two-object system is
Equation:

$$m_1v_1 + m_2v_2 = m_1v\prime_1 + m_2v\prime_2.$$

The only unknown in this equation is $v\prime_2$. Solving for $v\prime_2$ and substituting known values into the previous equation yields
Equation:

$$\begin{aligned} v\prime_2 &= \frac{m_1v_1 + m_2v_2 - m_1v\prime_1}{m_2} \\ &= \frac{(0.350\text{ kg})(2.00\text{ m/s}) + (0.500\text{ kg})(-0.500\text{ m/s})}{0.500\text{ kg}} - \frac{(0.350\text{ kg})(-4.00\text{ m/s})}{0.500\text{ kg}} \\ &= 3.70\text{ m/s.} \end{aligned}$$

Solution for (b)
The internal kinetic energy before the collision is
Equation:

$$\begin{aligned} \text{KE}_{\text{int}} &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\ &= \frac{1}{2}(0.350\text{ kg})(2.00\text{ m/s})^2 + \frac{1}{2}(0.500\text{ kg})(-0.500\text{ m/s})^2 \\ &= 0.763\text{ J.} \end{aligned}$$

After the collision, the internal kinetic energy is
Equation:

$$\begin{aligned} \text{KE}\prime_{\text{int}} &= \frac{1}{2}m_1v\prime_1^2 + \frac{1}{2}m_2v\prime_2^2 \\ &= \frac{1}{2}(0.350\text{ kg})(-4.00\text{ m/s})^2 + \frac{1}{2}(0.500\text{ kg})(3.70\text{ m/s})^2 \\ &= 6.22\text{ J.} \end{aligned}$$

The change in internal kinetic energy is thus
Equation:

$$\begin{aligned} \text{KE}\prime_{\text{int}} - \text{KE}_{\text{int}} &= 6.22\text{ J} - 0.763\text{ J} \\ &= 5.46\text{ J.} \end{aligned}$$

Discussion

The final velocity of cart 2 is large and positive, meaning that it is moving to the right after the collision. The internal kinetic energy in this collision increases by 5.46 J. That energy was released by the spring.

Section Summary

- An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).
- A collision in which the objects stick together is sometimes called perfectly inelastic because it reduces internal kinetic energy more than does any other type of inelastic collision.
- Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Conceptual Questions

Exercise:

Problem:

What is an inelastic collision? What is a perfectly inelastic collision?

Exercise:

Problem:

A small pickup truck that has a camper shell slowly coasts toward a red light with negligible friction. Two dogs in the back of the truck are moving and making various inelastic collisions with each other and the walls. What is the effect of the dogs on the motion of the center of mass of the system (truck plus entire load)? What is their effect on the motion of the truck?

Problems & Exercises

Exercise:

Problem:

A 0.240-kg billiard ball that is moving at 3.00 m/s strikes the bumper of a pool table and bounces straight back at 2.40 m/s (80% of its original speed). The collision lasts 0.0150 s. (a) Calculate the average force exerted on the ball by the bumper. (b) How much kinetic energy in joules is lost during the collision? (c) What percent of the original energy is left?

Solution:

(a) 86.4 N perpendicularly away from the bumper

(b) 0.389 J

(c) 64.0%

Exercise:

Problem: **Professional Application**

Two manned satellites approaching one another, at a relative speed of 0.250 m/s, intending to dock. The first has a mass of 4.00×10^3 kg, and the second a mass of 7.50×10^3 kg. (a) Calculate the final velocity (after docking) by using the frame of reference in which the first satellite was originally at rest. (b) What is the loss of kinetic energy in this inelastic collision? (c) Repeat both parts by using the frame of reference in which the second satellite was originally at rest. Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

Solution:

(a) 0.163 m/s in the direction of motion of the more massive satellite

(b) 81.6 J

(c) $8.70 \times 10^{-2}\ \text{m/s}$ in the direction of motion of the less massive satellite, 81.5 J. Because there are no external forces, the velocity of the center of mass of the two-satellite system is unchanged by the collision. The two velocities calculated above are the velocity of the center of mass in each of the two different individual reference frames. The loss in KE is the same in both reference frames because the KE lost to internal forces (heat, friction, etc.) is the same regardless of the coordinate system chosen.

Exercise:

Problem: Professional Application

A 30,000-kg freight car is coasting at 0.850 m/s with negligible friction under a hopper that dumps 110,000 kg of scrap metal into it. (a) What is the final velocity of the loaded freight car? (b) How much kinetic energy is lost?

Exercise:

Problem: Professional Application

One of the waste products of a nuclear reactor is plutonium-239 $\left({}^{239}\text{Pu}\right)$. This nucleus is radioactive and decays by splitting into a helium-4 nucleus and a uranium-235 nucleus $\left({}^{4}\text{He} + {}^{235}\text{U}\right)$, the latter of which is also radioactive and will itself decay some time later. The energy emitted in the plutonium decay is $8.40 \times 10^{-13}\ \text{J}$ and is entirely converted to kinetic energy of the helium and uranium nuclei. The mass of the helium nucleus is $6.68 \times 10^{-27}\ \text{kg}$, while that of the uranium is $3.92 \times 10^{-25}\ \text{kg}$ (note that the ratio of the masses is 4 to 235). (a) Calculate the velocities of the two nuclei, assuming the plutonium nucleus is originally at rest. (b) How much kinetic energy does each nucleus carry away? Note that the data given here are accurate to three digits only.

Exercise:

Problem: Professional Application

The Moon’s craters are remnants of meteorite collisions. Suppose a fairly large asteroid that has a mass of 5.00×10^{12} kg (about a kilometer across) strikes the Moon at a speed of 15.0 km/s. (a) At what speed does the Moon recoil after the perfectly inelastic collision (the mass of the Moon is 7.36×10^{22} kg) ? (b) How much kinetic energy is lost in the collision? Such an event may have been observed by medieval English monks who reported observing a red glow and subsequent haze about the Moon.

Solution:

(a) $1.02 \times 10^{-6}\ \mathrm{m/s}$

(b) $5.63 \times 10^{20}\ \mathrm{J}$ (almost all KE lost)

Exercise:

Problem:

What is the speed of a garbage truck that is 1.20×10^{4} kg and is initially moving at 25.0 m/s just after it hits and adheres to a trash can that is 80.0 kg and is initially at rest?

Solution:

24.8 m/s

Exercise:

Problem:

(a) During an ice skating performance, an initially motionless 80.0-kg clown throws a fake barbell away. The clown’s ice skates allow her to recoil frictionlessly. If the clown recoils with a velocity of 0.500 m/s and the barbell is thrown with a velocity of 10.0 m/s, what is the mass of the barbell? (b) How much kinetic energy is gained by this maneuver? (c) Where does the kinetic energy come from?

Solution:

(a) 4.00 kg

(b) 210 J

(c) The clown does work to throw the barbell, so the kinetic energy comes from the muscles of the clown. The muscles convert the chemical potential energy of ATP into kinetic energy.

Glossary

inelastic collision
: a collision in which internal kinetic energy is not conserved

perfectly inelastic collision
: a collision in which the colliding objects stick together

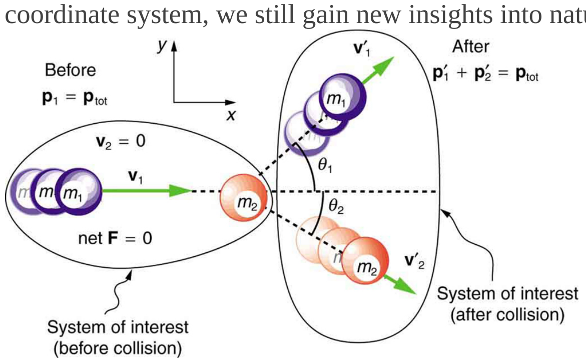
6.7 Collisions of Point Masses in Two Dimensions

- Discuss two dimensional collisions as an extension of one dimensional analysis.
- Define point masses.
- Derive an expression for conservation of momentum along x-axis and y-axis.
- Describe elastic collisions of two objects with equal mass.
- Determine the magnitude and direction of the final velocity given initial velocity, and scattering angle.

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. We will not consider such rotation until later, and so for now we arrange things so that no rotation is possible. To avoid rotation, we consider only the scattering of **point masses**—that is, structureless particles that cannot rotate or spin.

We start by assuming that $\mathbf{F}_{\text{net}} = 0$, so that momentum $\mathbf{p}$ is conserved. The simplest collision is one in which one of the particles is initially at rest. (See [link].) The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in [link]. Because momentum is conserved, the components of momentum along the x- and y-axes $(p_x \text{ and } p_y)$ will also be conserved, but with the chosen coordinate system, p_y is initially zero and p_x is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)



A two-dimensional collision with the coordinate system chosen so that m_2 is initially at rest and v_1 is parallel to the x -axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the x-axis, the equation for conservation of momentum is
Equation:

$$p_{1x} + p_{2x} = p\prime_{1x} + p\prime_{2x}.$$

Where the subscripts denote the particles and axes and the primes denote the situation after the collision. In terms of masses and velocities, this equation is
Equation:

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v\prime_{1x} + m_2 v\prime_{2x}.$$

But because particle 2 is initially at rest, this equation becomes
Equation:

$$m_1 v_{1x} = m_1 v\prime_{1x} + m_2 v\prime_{2x}.$$

The components of the velocities along the x-axis have the form v cos θ. Because particle 1 initially moves along the x-axis, we find $v_{1x} = v_1$.

Conservation of momentum along the x-axis gives the following equation:
Equation:

$$m_1 v_1 = m_1 v\prime_1 \text{ cos } \theta_1 + m_2 v\prime_2 \text{ cos } \theta_2,$$

where θ_1 and θ_2 are as shown in [link].

Note:
Conservation of Momentum along the x-axis
Equation:

$$m_1 v_1 = m_1 v\prime_1 \text{ cos } \theta_1 + m_2 v\prime_2 \text{ cos } \theta_2$$

Along the y-axis, the equation for conservation of momentum is
Equation:

$$p_{1y} + p_{2y} = p\prime_{1y} + p\prime_{2y}$$

or
Equation:

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v\prime_{1y} + m_2 v\prime_{2y}.$$

But v_{1y} is zero, because particle 1 initially moves along the x-axis. Because particle 2 is initially at rest, v_{2y} is also zero. The equation for conservation of momentum along the y-axis becomes
Equation:

$$0 = m_1 v\prime_{1y} + m_2 v\prime_{2y}.$$

The components of the velocities along the y-axis have the form $v\sin\theta$.

Thus, conservation of momentum along the y-axis gives the following equation:
Equation:

$$0 = m_1 v\prime_1 \sin\theta_1 + m_2 v\prime_2 \sin\theta_2.$$

Note:
Conservation of Momentum along the y-axis
Equation:

$$0 = m_1 v\prime_1 \sin\theta_1 + m_2 v\prime_2 \sin\theta_2$$

The equations of conservation of momentum along the x-axis and y-axis are very useful in analyzing two-dimensional collisions of particles, where one is originally stationary (a common laboratory situation). But two equations can only be used to find two unknowns, and so other data may be necessary when collision experiments are used to explore nature at the subatomic level.

Example:
Determining the Final Velocity of an Unseen Object from the Scattering of Another Object
Suppose the following experiment is performed. A 0.250-kg object (m_1) is slid on a frictionless surface into a dark room, where it strikes an initially stationary object with mass of 0.400 kg (m_2). The 0.250-kg object emerges from the room at an angle of 45.0º with its incoming direction.
The speed of the 0.250-kg object is originally 2.00 m/s and is 1.50 m/s after the collision. Calculate the magnitude and direction of the velocity $(v\prime_2$ and $\theta_2)$ of the 0.400-kg object after the collision.
Strategy
Momentum is conserved because the surface is frictionless. The coordinate system shown in [link] is one in which m_2 is originally at rest and the initial velocity is parallel to the x-axis, so that conservation of momentum along the x- and y-axes is applicable.
Everything is known in these equations except $v\prime_2$ and θ_2, which are precisely the quantities we wish to find. We can find two unknowns because we have two independent equations: the equations describing the conservation of momentum in the x- and y-directions.
Solution
Solving $m_1 v_1 = m_1 v\prime_1 \cos\theta_1 + m_2 v\prime_2 \cos\theta_2$ for $v'_2 \cos\theta_2$ and $0 = m_1 v\prime_1 \sin\theta_1 + m_2 v\prime_2 \sin\theta_2$ for $v\prime_2 \sin\theta_2$ and taking the ratio yields an equation (in which θ_2 is the only unknown quantity. Applying the identity $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$, we obtain:
Equation:

$$\tan\theta_2 = \frac{v\prime_1 \sin\theta_1}{v\prime_1 \cos\theta_1 - v_1}.$$

Entering known values into the previous equation gives
Equation:

$$\tan\theta_2 = \frac{(1.50\text{ m/s})(0.7071)}{(1.50\text{ m/s})(0.7071) - 2.00\text{ m/s}} = -1.129.$$

Thus,

Equation:

$$\theta_2 = \tan^{-1}(-1.129) = 311.5^{\circ} \approx 312^{\circ}.$$

Angles are defined as positive in the counter clockwise direction, so this angle indicates that m_2 is scattered to the right in [link], as expected (this angle is in the fourth quadrant). Either equation for the x- or y-axis can now be used to solve for $v\prime_2$, but the latter equation is easiest because it has fewer terms.

Equation:

$$v\prime_2 = -\frac{m_1}{m_2} v\prime_1 \frac{\sin\theta_1}{\sin\theta_2}$$

Entering known values into this equation gives

Equation:

$$v\prime_2 = -\left(\frac{0.250\ \text{kg}}{0.400\ \text{kg}}\right)(1.50\ \text{m/s})\left(\frac{0.7071}{-0.7485}\right).$$

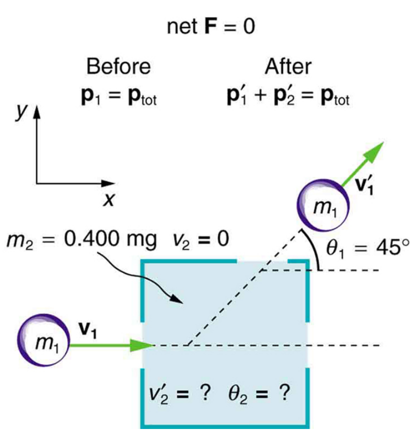
Thus,

Equation:

$$v\prime_2 = 0.886\ \text{m/s}.$$

Discussion

It is instructive to calculate the internal kinetic energy of this two-object system before and after the collision. (This calculation is left as an end-of-chapter problem.) If you do this calculation, you will find that the internal kinetic energy is less after the collision, and so the collision is inelastic. This type of result makes a physicist want to explore the system further.



A collision taking place in a dark room is explored in [link]. The incoming object m_1 is scattered by an initially stationary object. Only the stationary object's mass m_2 is known. By measuring the angle and speed at which m_1 emerges from the room, it is possible to calculate the magnitude and

direction of the initially stationary object’s velocity after the collision.

Elastic Collisions of Two Objects with Equal Mass

Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. This situation is nearly the case with colliding billiard balls, and precisely the case with some subatomic particle collisions. We can thus get a mental image of a collision of subatomic particles by thinking about billiards (or pool). (Refer to [link] for masses and angles.) First, an elastic collision conserves internal kinetic energy. Again, let us assume object 2 (m_2) is initially at rest. Then, the internal kinetic energy before and after the collision of two objects that have equal masses is

Equation:

$$\frac{1}{2}m{v_1}^2=\frac{1}{2}m{v\prime_1}^2+\frac{1}{2}m{v\prime_2}^2.$$

Because the masses are equal, $m_1=m_2=m$. Algebraic manipulation (left to the reader) of conservation of momentum in the x- and y-directions can show that

Equation:

$$\frac{1}{2}m{v_1}^2=\frac{1}{2}m{v\prime_1}^2+\frac{1}{2}m{v\prime_2}^2+mv\prime_1v\prime_2\cos(\theta_1-\theta_2).$$

(Remember that θ_2 is negative here.) The two preceding equations can both be true only if

Equation:

$$mv\prime_1v\prime_2\cos(\theta_1-\theta_2)=0.$$

There are three ways that this term can be zero. They are

- $v\prime_1=0$: head-on collision; incoming ball stops
- $v\prime_2=0$: no collision; incoming ball continues unaffected
- $\cos(\theta_1-\theta_2)=0$: angle of separation $(\theta_1-\theta_2)$ is 90º after the collision

All three of these ways are familiar occurrences in billiards and pool, although most of us try to avoid the second. If you play enough pool, you will notice that the angle between the balls is very close to 90º after the collision, although it will vary from this value if a great deal of spin is placed on the ball. (Large spin carries in extra energy and a quantity called *angular momentum*, which must also be conserved.) The assumption that the scattering of billiard balls is elastic is reasonable based on the correctness of the three results it produces. This assumption also implies that, to a good approximation, momentum is conserved for the two-ball system in billiards and pool. The problems below explore these and other characteristics of two-dimensional collisions.

Note:
Connections to Nuclear and Particle Physics
Two-dimensional collision experiments have revealed much of what we know about subatomic particles, as we shall see in Medical Applications of Nuclear Physics and Particle Physics. Ernest Rutherford, for example, discovered the nature of the atomic nucleus from such experiments.

Section Summary

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes. Choose a coordinate system with the x-axis parallel to the velocity of the incoming particle.
- Two-dimensional collisions of point masses where mass 2 is initially at rest conserve momentum along the initial direction of mass 1 (the x-axis), stated by $m_1 v_1 = m_1 v\prime_1 \cos\theta_1 + m_2 v\prime_2 \cos\theta_2$ and along the direction perpendicular to the initial direction (the y-axis) stated by $0 = m_1 v\prime_{1y} + m_2 v\prime_{2y}$.
- The internal kinetic before and after the collision of two objects that have equal masses is **Equation:**

$$\frac{1}{2}m{v_1}^2 = \frac{1}{2}m{v\prime_1}^2 + \frac{1}{2}m{v\prime_2}^2 + mv\prime_1 v\prime_2 \cos(\theta_1 - \theta_2).$$

- Point masses are structureless particles that cannot spin.

Problems & Exercises

Exercise:

Problem:

Two identical pucks collide on an air hockey table. One puck was originally at rest. (a) If the incoming puck has a speed of 6.00 m/s and scatters to an angle of 30.0^o,what is the velocity (magnitude and direction) of the second puck? (You may use the result that $\theta_1 - \theta_2 = 90^\text{o}$ for elastic collisions of objects that have identical masses.) (b) Confirm that the collision is elastic.

Solution:

(a) 3.00 m/s, 60^o below x-axis

(b) Find speed of first puck after collision:
$0 = mv\prime_1 \sin 30^\text{o} - mv\prime_2 \sin 60^\text{o} \Rightarrow v\prime_1 = v\prime_2 \frac{\sin 60^\text{o}}{\sin 30^\text{o}} = 5.196\ \text{m/s}$

Verify that ratio of initial to final KE equals one: $\left.\begin{array}{l}\text{KE} = \frac{1}{2}m{v_1}^2 = 18m\ \text{J} \\ \text{KE} = \frac{1}{2}m{v\prime_1}^2 + \frac{1}{2}m{v\prime_2}^2 = 18m\ \text{J}\end{array}\right\} \frac{\text{KE}}{\text{KE}\prime} = 1.00$

Exercise:

Problem:

Confirm that the results of the example [link] do conserve momentum in both the x- and y-directions.

Exercise:

Problem:

A 3000-kg cannon is mounted so that it can recoil only in the horizontal direction. (a) Calculate its recoil velocity when it fires a 15.0-kg shell at 480 m/s at an angle of 20.0^o above the horizontal. (b) What is the kinetic energy of the cannon? This energy is dissipated as heat transfer in shock absorbers that stop its recoil. (c) What happens to the vertical component of momentum that is imparted to the cannon when it is fired?

Solution:

(a) $-2.26\ \text{m/s}$

(b) $7.63 \times 10^3 \text{ J}$

(c) The ground will exert a normal force to oppose recoil of the cannon in the vertical direction. The momentum in the vertical direction is transferred to the earth. The energy is transferred into the ground, making a dent where the cannon is. After long barrages, cannon have erratic aim because the ground is full of divots.

Exercise:

Problem: **Professional Application**

Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei $\left(^{4}\text{He}\right)$ from gold-197 nuclei $\left(^{197}\text{Au}\right)$. The energy of the incoming helium nucleus was $8.00 \times 10^{-13} \text{ J}$, and the masses of the helium and gold nuclei were $6.68 \times 10^{-27} \text{ kg}$ and $3.29 \times 10^{-25} \text{ kg}$, respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of 120^{o} during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

Solution:

(a) $5.36 \times 10^5 \text{ m/s}$ at -29.5^{o}

(b) $7.52 \times 10^{-13} \text{ J}$

Exercise:

Problem:

Starting with equations $m_1 v_1 = m_1 v\prime_1 \cos \theta_1 + m_2 v\prime_2 \cos \theta_2$ and $0 = m_1 v\prime_1 \sin \theta_1 + m_2 v\prime_2 \sin \theta_2$ for conservation of momentum in the x- and y-directions and assuming that one object is originally stationary, prove that for an elastic collision of two objects of equal masses,

Equation:

$$\frac{1}{2} m {v_1}^2 = \frac{1}{2} m {v\prime_1}^2 + \frac{1}{2} m {v\prime_2}^2 + m v\prime_1 v\prime_2 \cos \left(\theta_1 - \theta_2\right)$$

as discussed in the text.

Solution:

We are given that $m_1 = m_2 \equiv m$. The given equations then become:

Equation:

$$v_1 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

and

Equation:

$$0 = v\prime_1 \sin \theta_1 + v\prime_2 \sin \theta_2.$$

Square each equation to get

Equation:

$$\begin{aligned} {v_1}^2 &= {v\prime_1}^2 \cos^2\theta_1 + {v\prime_2}^2 \cos^2\theta_2 + 2v\prime_1 v\prime_2 \cos\theta_1 \cos\theta_2 \\ 0 &= {v\prime_1}^2 \sin^2\theta_1 + {v\prime_2}^2 \sin^2\theta_2 + 2v\prime_1 v\prime_2 \sin\theta_1 \sin\theta_2. \end{aligned}$$

Add these two equations and simplify:
Equation:

$$\begin{aligned} {v_1}^2 &= {v\prime_1}^2 + {v\prime_2}^2 + 2v\prime_1 v\prime_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \\ &= {v\prime_1}^2 + {v\prime_2}^2 + 2v\prime_1 v\prime_2 \left[\frac{1}{2}\cos(\theta_1 - \theta_2) + \frac{1}{2}\cos(\theta_1 + \theta_2) + \frac{1}{2}\cos(\theta_1 - \theta_2) - \frac{1}{2}\cos(\theta_1 + \theta_2)\right] \\ &= {v\prime_1}^2 + {v\prime_2}^2 + 2v\prime_1 v\prime_2 \cos(\theta_1 - \theta_2). \end{aligned}$$

Multiply the entire equation by $\frac{1}{2}m$ to recover the kinetic energy:
Equation:

$$\frac{1}{2}m{v_1}^2 = \frac{1}{2}m{v\prime_1}^2 + \frac{1}{2}m{v\prime_2}^2 + mv\prime_1 v\prime_2 \cos(\theta_1 - \theta_2)$$

Exercise:

Problem: **Integrated Concepts**

A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

Glossary

point masses
: structureless particles with no rotation or spin